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APPROXIMATE THEORY FOR THE TRANSVERSE MOTION OF A TWO-LAYERED PLATE

A theoretical model applicable to acoustic scattering from submerged structures with coating layer(s)

B.Q. Vu

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

An approximate theory is developed for the transverse motion of a two-layered plate. The two layers are assumed to be elastic, isotropic, homogeneous, and welded to each other. The theory is similar to the Timoshenko-Mindlin model for a single-layered plate in which corrections are included for shear and rotary inertia. A single equation of transverse motion is derived that predicts the first four antisymmetric modes and is applicable to practical problems. Phase velocities of these modes are numerically analyzed for a plate consisting of a copper layer and a steel layer.

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OBJECTIVE

Develop an approximate model for the vibration of a two-layered plate. This model will be applied to the theoretical investigation of the scattering of acoustic waves from submerged structures with coating layer(s).

RESULTS

1. A single equation is developed for the transverse motion of a two-layered plate.

2. An equation is derived for the phase velocity of straight-crested waves.

3. Phase velocities of the first four antisymmetric modes of motion are calculated for an illustrative case of a steel layer and a copper layer.

RECOMMENDATIONS

1. Employ this model to investigate the behavior of the scattered signal (target strength) from a two-layered plate. A useful approach is to assume the plate to be loaded with water on one side and air or water on the other side, then to compute and plot the transmission and reflection coefficients of an incident acoustic wave. The procedure for performing the calculations is very similar to that of Graff et al. The variation of the coefficients versus the angle of incidence and the wavenumber-thickness product yields information about target strength and its variation at various modes of plate vibration. Furthermore, since the coefficients can be computed from the exact theory, a quantitative comparison can be made and the validity of this approximate model can be examined.

2. Apply this model to a composite plate of one elastic layer (base plate) and one viscoelastic layer (coating layer), to calculate target strength. Impedance discontinuities along the plate such as stiffening ribs and varying thickness can be handled by appropriately modifying the model.

3. Note that this model can also be used where the viscoelastic layer contains many inclusions and/or cavities, provided the "equivalent effective elastic properties" of the viscoelastic layer are calculated and used.

4. If experience with a single-layered plate is any indicator, this model should be improved to provide better resolution of the scattered signal of acoustic waves at grazing incident angles, possibly by combining it with the "Lyamshev Theory" on symmetric modes of vibration.

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CONTENTS

INTRODUCTION

When an acoustic wave strikes a plate of homogeneous, isotropic, and elastic material, the plate vibrates in two manners of motion: extensional and flexural, each containing infinite numbers of modes. In extensional motion, the displacement of the material particle, averaged over the thickness, is in the direction parallel to the plane of the plate. This also is termed symmetric (or longitudinal) motion, and the associated modes of vibration are known as symmetric modes. In flexural motion, the average displacement is transverse to the plane of the plate. This is also termed antisymmetric motion, and the associated modes of vibration are known as antisymmetric modes. As a result of the combination of these modes of vibration, any disturbance propagates along the plate at a speed that is not constant but depends on the relative ratio of wavelength and plate thickness. Knowledge of this dispersive characteristic of wave speed in plates is necessary in the study of wave scattering.

The exact solutions of problems of wave propagation in plates or layered plates of homogeneous, isotropic, and elastic materials of infinitely extended boundaries have been reported in the literature. The analysis is very complicated, and the solutions yield relationships between frequency and wavelength that yield an infinite number of dispersion curves. If the plates have an additional constraint such as a stiffener or a discontinuity in thickness or in material, it is impossible to obtain an exact solution. As a result, various plate theories have been established as means of approaching such problems.

The so-called classical theory of plate vibration is based on the Bernoulli-Euler theory of the bending of a beam when the cross section of the beam is assumed to remain plane and perpendicular to the neutral axis after bending. The classical theory does not describe the dispersion of the first antisymmetric mode of vibration and hence is accurate only for a thin plate at low frequencies of vibration. The Mindlin theory is superior when the rotary inertia of the cross section and the shear angle of the neutral plane are taken into account. In fact, the importance of rotary inertia and shear angle on the flexural motion of an isotropic, elastic, homogeneous beam was first investigated by Timoshenko (ref 1). Following Timoshenko, Mindlin (ref 2) formulated an approximate, two-dimensional theory of a vibrating plate in which he derived a single equation of motion involving transverse displacement. The phase velocity of straight-crested waves computed from his theory was seen to be in excellent agreement with the exact theory of elasticity.

For a two-layered plate, Jones (ref 3) used the exact theory to compute the phase velocity versus the thickness of the two layers. Lai (ref 4) introduced displacement fields, which include the shear and the rotary inertia corrections for the two layers. He obtained an analytical solution for the pressure radiation due to point and shear force excitations on the free faces of the plate and presented the numerical values of this solution.

[1] Timoshenko, SP, On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars, Philosophical Magazine, vol 41, p 744-746, 1921.

[2] Mindlin, RD, Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates, J of Applied Mech, vol 18, p 31-38, 1951.

[3] Jones, JP, Wave Propagation in a Two-Layered Medium, J of Applied Mech, vol 31, p 213-222, 1964.

[4] Lai, JL, Pressure Radiations from an Infinite Two-Layered Elastic Plate with Point and Shear Force Excitations, J of Acoustic Soc of America, vol 53, p 486-494, 1973.

Yu (ref 5) studied the sandwich plate, consisting of a core and two identical faces. He followed the analysis of Mindlin (ref 2) to formulate an approximate theory that includes corrections for shear and rotary inertia. In the limiting case where either the core approaches zero thickness or the two layers vanish, the sandwich plate becomes a single-layered plate and Yu's results become identical to those of Mindlin's work.

In many applications of acoustics, it is required to express the motion of a two-layered plate in a single equation of transverse displacement, but no such equation was found in the literature. This study is aimed at correcting that deficiency. The present analysis follows very closely analyses in references 2 and 5. Equations of motion in terms of transverse displacement and expressions of the phase velocity of straight-crested waves are developed for a plate consisting of two elastic, isotropic, and homogeneous layers of arbitrary thickness.

FORMULATION

Figure 1 shows a two-layered plate of thickness H. The two elastic, homogeneous, and isotropic layers of thickness h_1 and h_2 are labeled layer 1 and layer 2, each with Young's modulus E_i, Poisson's ratio ν_i, shear modulus μ_i, and density ρ_i. The coordinate system is shown with plane Oxy coincident with the interface and axis Oz normal to it. The interface contact is a weld and the layers are extended infinitely in plane Oxy. To simplify the involved algebra, the problem is restricted to the plane strain condition; only a unit length of the plate needs to be taken in the y-direction. The displacement in the y-direction and its derivatives are identically zero. The free faces of the plate are shown loaded with a fluid that exerts normal pressures p_{z1} and p_{z2} on opposite sides.

Following Mindlin (ref 2) we first define the equivalent "plate stress" terms and correlate them with the strain components, $\epsilon_{\alpha\beta}$, of each layer. Then we assume the displacement fields u_α such that they are continuous at the interface. Finally, we use the equations of motion in the theory of elasticity:

$$\sigma_{\alpha\beta,\beta} + \rho f_\alpha = \rho \ddot{u}_\alpha \ .$$

We substitute the previously assumed displacement fields and the expressions for plate stresses into these equations of motion to obtain the equations of motion for our layered plate. These new equations of motion correlate the plate stresses and the assumed displacement fields. By using the plate stress-strain relationships, we obtain a set of equations of motion that relate the assumed displacement fields and the plate's elastic properties.

5 Yu, YY, A New Theory of Elastic Sandwich Plates—One Dimensional Case, J of Applied Mech, vol 26, p 415-421, 1959.
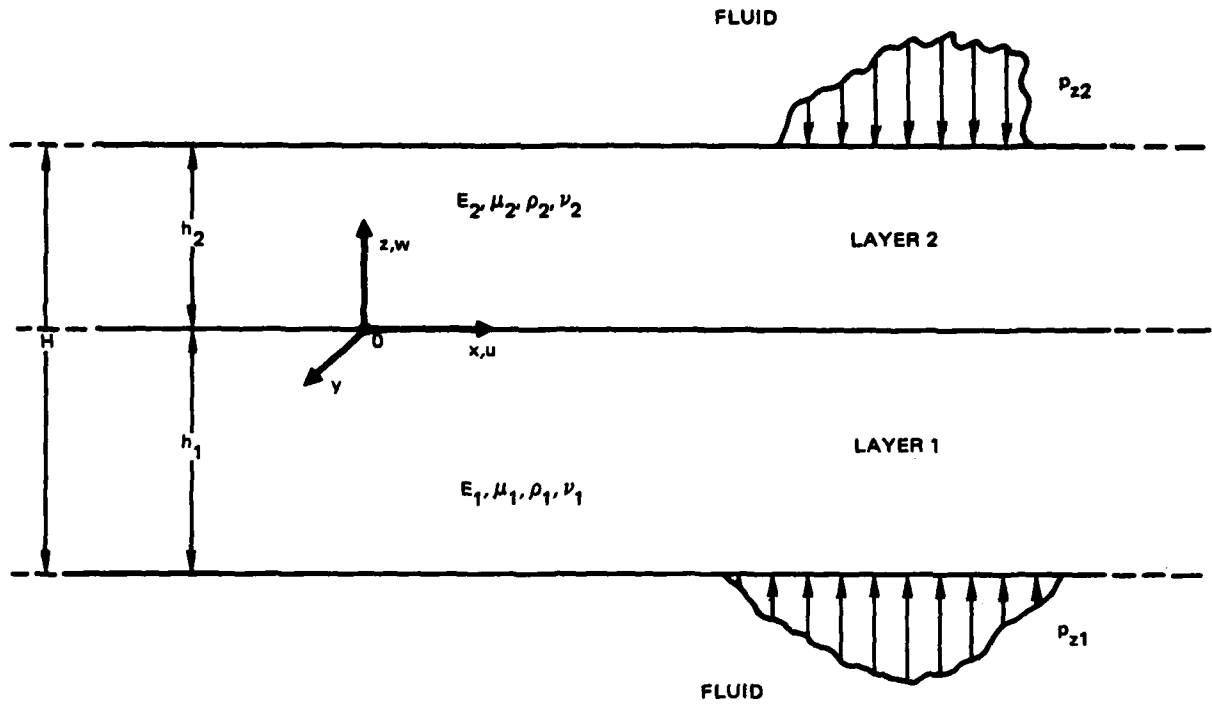
Figure 1. Geometry of the two-layered plate.

PLATE STRESS COMPONENTS

The bending moments, in-plane forces, and transverse shearing forces are defined respectively as follows:

$$M_{x1} = \int_{-h_1}^{0} \sigma_{xx1} z \, dz \, , \quad M_{x2} = \int_{0}^{h_2} \sigma_{xx2} z \, dz$$

$$Q_{x1} = \int_{-h_1}^{0} \tau_{zx1} \, dz \, , \quad Q_{x2} = \int_{0}^{h_2} \tau_{zx2} \, dz \qquad , \tag{1}$$

$$N_{x1} = \int_{-h_1}^{0} \sigma_{xx1} \, dz \, , \quad N_{x2} = \int_{0}^{h_2} \sigma_{xx2} \, dz$$

where the subscripts 1 and 2 indicate layer 1 and layer 2.

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From Hooke's law:

$$\sigma_{\alpha\beta} = \lambda e_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu e_{\alpha\beta} \ .$$

We can write the stress-strain relationship as follows:

$$\sigma_{xxi} = \left(\lambda_i + 2\mu_i \right) e_{xxi} + \lambda e_{zzi}$$

$$\sigma_{zzi} = \left(\lambda_i + 2\mu_i \right) e_{zzi} + \lambda e_{xxi} \qquad , \tag{2}$$

$$\tau_{xzi} = \mu_i \gamma_{xzi}$$

where λ_i and μ_i are Lamé's constants and are given by the relationships

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \ ; \ \mu = \frac{E}{2(1+\nu)} \ .$$

Eliminating e_{zz} from (2) to obtain the stress-strain relationship:

$$\sigma_{xxi} = \frac{E_i}{1-\nu_i^2} e_{xxi} + \frac{\lambda_i}{\lambda_i + 2\mu_i} \sigma_{zzi}$$

$$\tag{3}$$

$$\tau_{xzi} = \mu_i \gamma_{xzi}$$

Substitution of (3) into (1) yields the following:

$$M_{xi} = \int_{a_i}^{b_i} \frac{E_i}{\left(1-\nu_i^2\right)} e_{xxi} z \, dz + \int_{a_i}^{b_i} \frac{\lambda_i}{\lambda_i + 2\mu_i} \sigma_{zzi} z \, dz$$

$$Q_{xi} = \int_{a_i}^{b_i} \mu_i \gamma_{xzi} \qquad . \tag{4}$$

$$N_{xi} = \int_{a_i}^{b_i} \frac{E_i}{1-\nu_i^2} e_{xxi} \, dz + \int_{a_i}^{b_i} \frac{\lambda_i}{\lambda_i + 2\mu_i} \sigma_{zzi} \, dz$$

Since there is no concentrating force acting on the plate in the z-direction, σ_{zzi}, in general, are small. Consequently the integrals containing σ_{zzi} are negligible compared to those of the in-plane stresses σ_{xxi}. It is therefore assumed that these integrals are equal to zero, which leads to the following relations:

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$$M_{xi} = \frac{E_i}{1-\nu_i^2} \int_{a_i}^{b_i} \epsilon_{xxi} z \, dz$$

$$Q_{xi} = \kappa \mu_i \int_{a_i}^{b_i} \gamma_{xzi} \, dz \qquad , \qquad (5)$$

$$N_{xi} = \frac{E_i}{1-\nu_i^2} \int_{a_i}^{b_i} \epsilon_{xxi} \, dz$$

where κ is the shear correction, introduced exactly as in Mindlin (ref 2). κ assumes the value of $\pi^2/12$. (Note that Mindlin used κ^2 instead of κ.)

PLATE DISPLACEMENT COMPONENTS

Following Mindlin and Medick (ref 6) and Yu (ref 5), the displacement fields can be written as follows:

$$u_i(x, z, t) = u_i^{(0)}(x, t) + z \, u_i^{(1)}(x, t)$$

$$, \qquad (6)$$

$$w_i(x, z, t) = w(x, t)$$

where the superscripts (0) and (1) denote the order of displacement and the subscript i denotes, as before, layer 1 and layer 2. $u_i^{(1)}$ represents the slope of the cross section of the layers with respect to the z-direction. In writing the above displacement fields we made two assumptions: the cross section of each layer remains plane during bending; and the thickness modes (stretch and shear) are eliminated, by setting $w_i = w(x, t)$. Note that if the shear correction were neglected, as in the classical plate theory, we would have

$$u_i^{(1)}(x, t) = -\frac{\partial}{\partial x} w(x, t).$$

The displacement fields are now written as follows:

$$u_1(x, z, t) = \psi_1(x, t) + z \, \psi_2(x, t)$$

$$u_2(x, z, t) = \psi_1(x, t) + z \, \psi_3(x, t) \qquad , \qquad (7)$$

$$w_1 = w_2(x, z, t) = w(x, t)$$

where ψ_1, ψ_2, ψ_3, and w are unknown functions. Notice that (7) satisfies the condition of continuity of displacement at the interface ($z = 0$).

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[6] Mindlin, RD, and MA Medick, Extensional Vibrations of Elastic Plates, J of Applied Mech, vol 26, p 561-569, 1959.

By substituting (7) into the strain-displacement relationship

$$\epsilon_{\alpha\beta} = \frac{1}{2} (u_{\alpha,\beta} + u_{\beta,\alpha}) \, ,$$

we obtain

$$\epsilon_{xx1} = \frac{\partial u_1}{\partial x} = \psi'_1 + z \, \psi'_2$$

$$\epsilon_{xx2} = \frac{\partial u_2}{\partial x} = \psi'_1 + z \, \psi'_3$$

$$\gamma_{xz1} = \frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x} = \psi_2 + w'$$

$$\gamma_{xz2} = \frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x} = \psi_3 + w'$$

$$(8)$$

where prime ($'$) denotes differentiation with respect to x. Inserting (8) into (5) yields the following:

$$M_{x1} = \frac{E_1}{1-\nu_1^2} \left(- \frac{h_1^2}{2} \, \psi'_1 + \frac{h_1^3}{3} \, \psi'_2 \right) = -D_1 \, \psi'_1 + \frac{2}{3} \, h_1 \, D_1 \, \psi'_2$$

$$M_{x2} = \frac{E_2}{1-\nu_2^2} \left(\frac{h_2^2}{2} \, \psi'_1 + \frac{h_2^3}{3} \, \psi'_3 \right) = D_2 \, \psi'_1 + \frac{2}{3} \, h_2 \, D_2 \, \psi'_3$$

$$N_{x1} = \frac{E_1}{1-\nu_1^2} \left(h_1 \, \psi'_1 - \frac{h_1^2}{2} \, \psi'_2 \right) = \frac{2}{h_1} \, D_1 \, \psi'_1 - D_1 \, \psi'_2$$

$$N_{x2} = \frac{E_2}{1-\nu_2^2} \left(h_2 \, \psi'_1 + \frac{h_2^2}{2} \, \psi'_3 \right) = \frac{2}{h_2} \, D_2 \, \psi'_1 + D_2 \, \psi'_3$$

$$(9)$$

$$Q_{x1} = \kappa \, \mu_1 \, h_1 \, (w' + \psi'_2) = G_1 \, (w' + \psi'_2)$$

$$Q_{x2} = \kappa \, \mu_2 \, h_2 \, (w' + \psi'_3) = G_2 \, (w' + \psi'_3)$$

where

$$D_1 = \frac{E_1}{1-\nu_1^2} \frac{h_1^2}{2} \quad , \quad D_2 = \frac{E_2}{1-\nu_2^2} \frac{h_2^2}{2}$$

$$G_1 = \kappa \mu_1 h_1 \quad , \quad G_2 = \kappa \mu_2 h_2 \; .$$

EQUATIONS OF MOTION

The equations of motion in the theory of elasticity,

$$\sigma_{\alpha\beta,\beta} + \rho f_\alpha = \rho \ddot{u}_\alpha \; ,$$

(where the dot ($\cdot$) represents differentiation with respect to time t) can be rewritten for the present case, omitting body forces, as follows:

$$\frac{\partial \sigma_{xxi}}{\partial x} + \frac{\partial \tau_{xzi}}{\partial z} - \rho_i \frac{\partial^2 u_i}{\partial t^2} = 0 \tag{10a}$$

$$\frac{\partial \sigma_{zzi}}{\partial z} + \frac{\partial \tau_{xzi}}{\partial x} - \rho_i \frac{\partial^2 w}{\partial t^2} = 0 \tag{10b}$$

Multiplying (10a) by z and integrating over the thickness, we obtain, for layer 1,

$$\int_{-h_1}^{0} \frac{\partial}{\partial x} \sigma_{xx1} z \, dz + \int_{-h_1}^{0} \frac{\partial}{\partial z} \tau_{xz1} z \, dz - \rho_1 \int_{-h_1}^{0} \frac{\partial^2 u_1}{\partial t^2} z \, dz = 0 \; . \tag{11}$$

The first term can be readily recognized as $\dfrac{\partial}{\partial x} M_{x1}$; the second term is integrated by part,

yielding

$$\int_{-h_1}^{0} \frac{\partial}{\partial z} \tau_{xz1} z \, dz = \left[z \, \tau_{xz1} \right]_{-h_1}^{0} - Q_{x1} \; ;$$

and the third term is rewritten as

$$\rho_1 \int_{-h_1}^{0} \ddot{u}_1 z \, dz = \rho_1 \int_{-h_1}^{0} \ddot{\psi}_1 z \, dz + \rho_1 \int_{-h_1}^{0} \ddot{\psi}_2 z^2 \, dz = -\frac{\rho_1 h_1^2}{2} \ddot{\psi}_1$$

$$+ \frac{\rho_1 h_1^3}{3} \ddot{\psi}_2 \; .$$

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Hence, (11) becomes

$$M'_{x1} - Q_{x1} + \left[z\,\tau_{xz1} \right]_{-h_1}^{0} + \frac{\rho_1 h_1^2}{2}\ddot{\psi}_1 - \frac{\rho_1 h_1^3}{3}\ddot{\psi}_2 = 0 \ . \tag{12}$$

Similarly for layer 2, we have

$$M'_{x2} - Q_{x2} + \left[z\,\tau_{xz2} \right]_{0}^{h_2} - \frac{\rho_2 h_2^2}{2}\ddot{\psi}_1 - \frac{\rho_2 h_2^3}{3}\ddot{\psi}_3 = 0 \ . \tag{13}$$

Note that the last terms in (12) and (13) represent the rotary inertia of the cross section of each layer.

When (10a) is again integrated over the thickness, we obtain, for layer 1,

$$N'_{x1} + \left[\tau_{xz1} \right]_{-h_1}^{0} - \rho_1 h_1 \ddot{\psi}_1 + \frac{\rho_1 h_1^2}{2}\ddot{\psi}_2 = 0 \ , \tag{14}$$

and for layer 2,

$$N'_{x2} + \left[\tau_{xz2} \right]_{0}^{h_2} - \rho_2 h_2 \ddot{\psi}_1 - \frac{\rho_2 h_2^2}{2}\ddot{\psi}_3 = 0 \ . \tag{15}$$

(10b) is integrated over the thickness, yielding, for layer 1,

$$Q'_{x1} + \left[\sigma_{zz1} \right]_{-h_1}^{0} - \rho_1 h_1 \ddot{w} = 0 \ , \tag{16}$$

and for layer 2,

$$Q'_{x2} + \left[\sigma_{zz2} \right]_{0}^{h_2} - \rho_2 h_2 \ddot{w} = 0 \ . \tag{17}$$

The boundary conditions of this problem are as follows:

At $z = -h_1$: $\sigma_{zz1} = -p_{z1}$, $\tau_{xz1} = 0$.

At $z = h_2$: $\sigma_{zz2} = -p_{z2}$, $\tau_{xz2} = 0$.

At $z = 0$: $\tau_{xz1} = \tau_{xz2}$, $\sigma_{zz1} = \sigma_{zz2}$.

Applying these into (12) and (13) yields

$$M'_{x1} - Q_{x1} + \frac{\rho_1 h_1^2}{2} \ddot{\psi}_1 - \frac{\rho_1 h_1^3}{3} \ddot{\psi}_2 = 0 . \tag{18}$$

$$M'_{x2} - Q_{x2} - \frac{\rho_2 h_2^2}{2} \ddot{\psi}_1 - \frac{\rho_2 h_2^3}{3} \ddot{\psi}_3 = 0 . \tag{19}$$

We now apply the boundary conditions into (14), (15), (16), and (17), adding (14) and (15) to obtain

$$N'_x - (\rho_1 h_1 + \rho_2 h_2) \ddot{\psi}_1 + \frac{\rho_1 h_1^2}{2} \ddot{\psi}_2 - \frac{\rho_2 h_2^2}{2} \ddot{\psi}_3 = 0 \tag{20}$$

and adding (16) and (17) to obtain

$$Q'_x + p - (\rho_1 h_1 + \rho_2 h_2) \ddot{w} = 0 , \tag{21}$$

where

$$N_x = N_{x1} + N_{x2}$$

$$Q_x = Q_{x1} + Q_{x2}$$

$$p = p_{z1} - p_{z2} .$$

(18), (19), (20), and (21) are the equations of motion involving the plate stresses and the assumed displacement fields.

Substituting the expressions for plate stress components, (9), into (18) - (21) yields the following:

$$-D_1 \psi''_1 + \frac{2}{3} h_1 D_1 \psi''_2 - G_1 (w' + \psi_2) = \frac{\rho_1 h_1^3}{3} \ddot{\psi}_2 - \frac{\rho_1 h_1^2}{2} \ddot{\psi}_1 \tag{22}$$

$$D_2 \psi_1'' + \frac{2}{3} h_2 D_2 \psi_3'' - G_2(w' + \psi_3) = \frac{\rho_2 h_2^3}{3} \ddot{\psi}_3 + \frac{\rho_2 h_2^2}{2} \ddot{\psi}_1 \tag{23}$$

$$\left(\frac{2}{h_1} D_1 + \frac{2}{h_2} D_2 \right) \psi_1'' - D_1 \psi_2'' + D_2 \psi_3'' = \frac{\rho_2 h_2^2}{2} \ddot{\psi}_3 - \frac{\rho_1 h_1^2}{2} \ddot{\psi}_2$$

$$+ (\rho_1 h_1 + \rho_2 h_2) \ddot{\psi}_1 \tag{24}$$

$$(G_1 + G_2) w'' + G_1 \psi_2' + G_2 \psi_3' + p = (\rho_1 h_1 + \rho_2 h_2) \ddot{w} \ . \tag{25}$$

(22) - (25) are the equations of motion connecting the assumed displacement fields and the elastic properties of the two layers; they are equivalent to equation 16 of Mindlin (ref 2) for a single-layered plate. In practical applications, it is sometimes required that the equations of motion be expressed as a single equation of $w(x, t)$. This can be done by eliminating ψ_1, ψ_2, and ψ_3 between (22), (23), and (24) and substituting into (25). These steps are performed next.

RESULTS

EQUATION OF TRANSVERSE MOTION

(22), (23), and (24) are rewritten as

$$L_1 \psi_1 + L_2 \psi_2 \ = L_{11} w \tag{22'}$$

$$L_4 \psi_1 + L_6 \psi_3 \ = L_{12} w \tag{23'}$$

$$L_7 \psi_1 + L_8 \psi_2 + L_9 \psi_3 \ = 0 \quad , \tag{24'}$$

where

$$L_1 \ = -D_1 \nabla^2 + \frac{\rho_1 h_1^2}{2} \frac{\partial^2}{\partial t^2}$$

$$L_2 \ = \frac{2}{3} h_1 D_1 \nabla^2 - G_1 - \frac{\rho_1 h_1^3}{3} \frac{\partial^2}{\partial t^2}$$

$$L_4 \ = D_2 \nabla^2 - \frac{\rho_2 h_2^2}{2} \frac{\partial^2}{\partial t^2}$$

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$$L_6 = \frac{2}{3} h_2 D_2 \nabla^2 - G_2 - \frac{\rho_2 h_2{}^3}{3} \frac{\partial^2}{\partial t^2}$$

$$L_7 = 2\left(\frac{D_1}{h_1} + \frac{D_2}{h_2}\right) \nabla^2 - (\rho_1 h_1 + \rho_2 h_2) \frac{\partial^2}{\partial t^2}$$

$$L_8 = -D_1 \nabla^2 + \frac{\rho_1 h_1{}^2}{2} \frac{\partial^2}{\partial t^2} = L_1$$

$$L_9 = D_2 \nabla^2 - \frac{\rho_2 h_2{}^2}{2} \frac{\partial^2}{\partial t^2} = L_4$$

$$L_{11} = G_1 \nabla$$

$$L_{12} = G_2 \nabla \quad .$$

Hence we have

$$\left[\mathcal{L}\right] \psi_2 = \text{Det} \begin{pmatrix} L_1 & L_{11}w & 0 \\ L_4 & L_{12}w & L_6 \\ L_7 & 0 & L_9 \end{pmatrix}$$

and

$$\left[\mathcal{L}\right] \psi_3 = \text{Det} \begin{pmatrix} L_1 & L_2 & L_{11}w \\ L_4 & 0 & L_{12}w \\ L_7 & L_8 & 0 \end{pmatrix} \quad ,$$

where

$$\left[\mathcal{L}\right] = \text{Det} \begin{pmatrix} L_1 & L_2 & 0 \\ L_4 & 0 & L_6 \\ L_7 & L_8 & L_9 \end{pmatrix} \quad .$$

Substituting the above into (25) yields

$$(G_1 + G_2) \nabla^2 \left[\mathcal{L} \right] w + G_1 \nabla \begin{pmatrix} L_1 & L_{11}w & 0 \\ L_4 & L_{12}w & L_6 \\ L_7 & 0 & L_9 \end{pmatrix} + G_2 \nabla \begin{pmatrix} L_1 & L_2 & L_{11}w \\ L_4 & 0 & L_{12}w \\ L_7 & L_8 & 0 \end{pmatrix}$$

$$+ \left[\mathcal{L} \right] p - (\rho_1 h_1 + \rho_2 h_2) \left[\mathcal{L} \right] \frac{\partial^2}{\partial t^2} w = 0 . \tag{26}$$

After expanding the determinants and rearranging the terms of like order, we obtain the equation of motion in terms of w(x, t):

$$\left\{ L_1 L_4 \left[\frac{2}{9} \nabla^2 (h_2 L_1 G_1 + h_2 L_1 G_2 - h_1 L_4 G_1 - h_1 L_4 G_2) \right. \right.$$

$$+ G_1 G_2 \nabla^2 \left(\frac{4}{3} \frac{h_1}{h_2} + \frac{4}{3} \frac{h_2}{h_1} + 2 \right) - \frac{2}{9} \rho \frac{\partial^2}{\partial t^2} (h_2 L_1 - h_1 L_4) \tag{27}$$

$$\left. - \frac{4}{3} \rho \frac{\partial^2}{\partial t^2} \left(\frac{h_1}{h_2} G_2 + \frac{h_2}{h_1} G_1 \right) \right] + L_1^2 \left(\frac{1}{3} G_2 \rho \frac{\partial^2}{\partial t^2} - \frac{1}{3} G_1 G_2 \nabla^2 \right)$$

$$\left. + L_4^2 \left(\frac{1}{3} G_1 \rho \frac{\partial^2}{\partial t^2} - \frac{1}{3} G_1 G_2 \nabla^2 \right) - 2 G_1 G_2 \rho \frac{\partial^2}{\partial t^2} \left(\frac{L_4}{h_2} - \frac{L_1}{h_1} \right) \right\} w + \left[\mathcal{L} \right] p = 0,$$

where

$$\rho = \rho_1 h_1 + \rho_2 h_2$$

and

$$\left[\mathcal{L} \right] = L_1 L_4 \left[\frac{2}{9} \left(h_2 L_1 - h_1 L_4 \right) + \frac{4}{3} \left(\frac{h_1}{h_2} G_2 + \frac{h_2}{h_1} G_1 \right) \right]$$

$$- \frac{1}{3} (G_2 L_1^2 + G_1 L_4^2) + 2 G_1 G_2 \left(\frac{L_4}{h_2} - \frac{L_1}{h_1} \right) .$$

(27) is the single equation of motion of w(x, t) and equivalent to equation 37 of Mindlin (ref 2).

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When the two layers are identical,

$$G_1 = G_2$$

$$L_1 = L_4$$

$$\rho_1 = \rho_2$$

$$h_1 = h_2 \; .$$

After some algebraic operations, (27) becomes

$$\left[L_1 \left(\frac{8}{9} \rho_1 h_1^2 \frac{\partial^2}{\partial t^2} - \frac{8}{9} h_1 G_1 \nabla^2 \right) + \frac{4}{3} G_1 \rho_1 h_1 \frac{\partial^2}{\partial t^2} \right] w$$

$$- \left(L_1 + \frac{3 G_1}{2 h_1} \right) \frac{4 h_1}{9} p = 0 \; . \tag{28}$$

(28) is the equation of motion for a single-layered plate; it becomes identical to equation 37 of Mindlin (ref 2) if $h/2$ is substituted for h_1. We substitute the expressions of L_1 and L_4 into (27), expand it, and perform some algebraic operations to yield the complete form of the equation of motion:

$$\left[\nabla^8 \left\{ \frac{2}{9} D_1 D_2 (D_1 h_2 + D_2 h_1)(G_1 + G_2) \right\} \right.$$

$$+ \nabla^6 \left\{ -\frac{1}{3} G_1 G_2 (D_1^2 + D_2^2) - \frac{2}{9} D_1 D_2 \frac{\partial^2}{\partial t^2} \left[\left(\frac{\rho_1 h_1^2 h_2}{2} + \frac{\rho_2 h_2^2 h_1}{2} \right) (G_1 + G_2) \right. \right.$$

$$\left. + (\rho_1 h_1 + \rho_2 h_2)(D_1 h_2 + D_2 h_1) \right]$$

$$- \frac{2}{9} \frac{\partial^2}{\partial t^2} \left(D_1 \frac{\rho_2 h_2^2}{2} + D_2 \frac{\rho_1 h_1^2}{2} \right) (D_1 h_2 + D_2 h_1)(G_1 + G_2)$$

$$- D_1 D_2 \left(\frac{4}{3} r_1 + \frac{4}{3} r_2 + 2 \right) G_1 G_2 \right\}$$

$$+ \nabla^4 \left\{ \frac{2}{9} \frac{\partial^4}{\partial t^4} \left[D_1 D_2 (\rho_1 h_1 + \rho_2 h_2) \left(\frac{\rho_1 h_1^2 h_2}{2} + \frac{\rho_2 h_2^2 h_1}{2} \right) \right. \right.$$

$$+ \left(D_1 \frac{\rho_2 h_2{}^2}{2} + D_2 \frac{\rho_1 h_1{}^2}{2}\right)\left(\frac{\rho_1 h_1{}^2 h_2}{2} + \frac{\rho_2 h_2{}^2 h_1}{2}\right)(G_1 + G_2)$$

$$+ \left(D_1 \frac{\rho_2 h_2{}^2}{2} + D_2 \frac{\rho_1 h_1{}^2}{2}\right)(\rho_1 h_1 + \rho_2 h_2)(D_1 h_2 + D_2 h_1)$$

$$+ \frac{\rho_1 h_1{}^2 \rho_2 h_2{}^2}{4} (D_1 h_2 + D_2 h_1)(G_1 + G_2) \Bigg]$$

$$+ \frac{\partial^2}{\partial t^2} \Bigg[\frac{4}{3} D_1 D_2 (\rho_1 h_1 + \rho_2 h_2)(r_1 G_2 + r_2 G_1)$$

$$+ \left(D_1 \frac{\rho_2 h_2{}^2}{2} + D_2 \frac{\rho_1 h_1{}^2}{2}\right)\left(\frac{4}{3}r_1 + \frac{4}{3}r_2 + 2\right) G_1 G_2$$

$$+ \frac{1}{3}(\rho_1 h_1 + \rho_2 h_2)(D_1{}^2 G_2 + D_2{}^2 G_1)$$

$$+ \frac{1}{3} G_1 G_2 (D_1 \rho_1 h_1{}^2 + D_2 \rho_2 h_2{}^2) \Bigg] \Bigg\}$$

$$+ \nabla^2 \Bigg\{ -\frac{2}{9} \frac{\partial^6}{\partial t^6} \Bigg[\left(D_1 \frac{\rho_2 h_2{}^2}{2} + D_2 \frac{\rho_1 h_1{}^2}{2}\right)(\rho_1 h_1 + \rho_2 h_2)\left(\frac{\rho_1 h_1{}^2 h_2}{2} + \frac{\rho_2 h_2{}^2 h_1}{2}\right)$$

$$+ \frac{\rho_1 \rho_2 h_1{}^2 h_2{}^2}{4}\left(\frac{\rho_1 h_1{}^2 h_2}{2} + \frac{\rho_2 h_2{}^2 h_1}{2}\right)(G_1 + G_2)$$

$$+ \frac{\rho_1 \rho_2 h_1{}^2 h_2{}^2}{4} (\rho_1 h_1 + \rho_2 h_2)(D_1 h_2 + D_2 h_1) \Bigg]$$

$$- \frac{\partial^4}{\partial t^4} \Bigg[\frac{4}{3}\left(D_1 \frac{\rho_2 h_2{}^2}{2} + D_2 \frac{\rho_1 h_1{}^2}{2}\right)(\rho_1 h_1 + \rho_2 h_2)(r_1 G_2 + r_2 G_1)$$

$$+ \frac{\rho_1 \rho_2 h_1{}^2 h_2{}^2}{4}\left(\frac{4}{3}r_1 + \frac{4}{3}r_2 + 2\right) G_1 G_2$$

$$+ \frac{1}{3} G_1 G_2 \left(\left\{\frac{\rho_1 h_1{}^2}{2,}\right\}^2 + \left\{\frac{\rho_2 h_2{}^2}{2}\right\}^2\right)$$

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$$+ \frac{1}{3} (\rho_1 h_1 + \rho_2 h_2) (G_2 D_1 \rho_1 h_1{}^2 + G_1 D_2 \rho_2 h_2{}^2) \Big]$$

$$- \frac{\partial^2}{\partial t^2} \left[2 G_1 G_2 (\rho_1 h_1 + \rho_2 h_2) \left(\frac{D_1}{h_1} + \frac{D_2}{h_2} \right) \right] \Bigg\}$$

$$+ \Bigg\{ \frac{1}{36} \frac{\partial^8}{\partial t^8} (\rho_1 \rho_2 h_1{}^2 h_2{}^2) (\rho_1 h_1 + \rho_2 h_2) (\rho_1 h_1{}^2 h_2 + \rho_2 h_2{}^2 h_1)$$

$$+ \frac{\partial^6}{\partial t^6} \left[\frac{1}{3} \rho_1 \rho_2 h_1{}^2 h_2{}^2 (\rho_1 h_1 + \rho_2 h_2) (r_1 G_2 + r_2 G_1) \right.$$

$$+ \frac{1}{3} (\rho_1 h_1 + \rho_2 h_2) \left. \left(G_2 \left\{ \frac{\rho_1 h_1{}^2}{2} \right\}^2 + G_1 \left\{ \frac{\rho_2 h_2{}^2}{2} \right\}^2 \right) \right]$$

$$+ \frac{\partial^4}{\partial t^4} \left[G_1 G_2 (\rho_1 h_1 + \rho_2 h_2)^2 \right] \Bigg\} \Bigg] \Bigg] \, w \quad + \left[\mathfrak{L} \right] p = 0 \, , \tag{29}$$

with

$$\left[\mathfrak{L} \right] = \nabla^6 \left\{ \frac{2}{9} D_1 D_2 (h_2 D_1 + h_1 D_2) \right\}$$

$$+ \nabla^4 \left\{ - \frac{2}{9} \frac{\partial^2}{\partial t^2} \left[D_1 D_2 \left(\frac{\rho_1 h_1{}^2 h_2}{2} + \frac{\rho_2 h_2{}^2 h_1}{2} \right) \right. \right.$$

$$+ \left. \left(D_1 \frac{\rho_2 h_2{}^2}{2} + D_2 \frac{\rho_1 h_1{}^2}{2} \right) (h_2 D_1 + h_1 D_2) \right] - \frac{4}{3} D_1 D_2 (r_1 G_2 + r_2 G_1) \Bigg\}$$

$$+ \nabla^2 \left\{ \frac{2}{9} \frac{\partial^4}{\partial t^4} \left[\left(D_1 \frac{\rho_2 h_2{}^2}{2} + D_2 \frac{\rho_1 h_1{}^2}{2} \right) \left(\frac{\rho_1 h_1{}^2 h_2}{2} + \frac{\rho_2 h_2{}^2 h_1}{2} \right) \right. \right.$$

$$+ \left. \left(\frac{\rho_1 \rho_2 h_1{}^2 h_2{}^2}{4} \right) (h_2 D_1 + h_1 D_2) \right]$$

$$+ \frac{\partial^2}{\partial t^2} \left[\frac{4}{3} \left(D_1 \frac{\rho_2 h_2{}^2}{2} + D_2 \frac{\rho_1 h_1{}^2}{2} \right) (r_1 G_2 + r_2 G_1) \right.$$

$$+ \frac{1}{3} \left(D_1 G_2 \rho_1 h_1{}^2 + D_2 G_1 \rho_2 h_2{}^2 \right) \Bigg] + 2 G_1 G_2 \left(\frac{D_2}{h_2} + \frac{D_1}{h_1} \right) \Bigg\}$$

$$+ \Bigg\{ -\frac{1}{36} \rho_1 \rho_2 h_1{}^2 h_2{}^2 (\rho_1 h_1{}^2 h_2 + \rho_2 h_2{}^2 h_1) \frac{\partial^6}{\partial t^6}$$

$$- \frac{\partial^4}{\partial t^4} \left[\frac{1}{3} \rho_1 \rho_2 h_1{}^2 h_2{}^2 (r_1 G_2 + r_2 G_1) \right.$$

$$+ \frac{1}{3} G_2 \left(\frac{\rho_1 h_1{}^2}{2} \right)^2 + \frac{1}{3} G_1 \left(\frac{\rho_2 h_2{}^2}{2} \right)^2 \Bigg]$$

$$- 2 G_1 G_2 \ (\rho_1 h_1 + \rho_2 h_2) \ \frac{\partial^2}{\partial t^2} \Bigg\} \quad .$$

EQUATION OF PHASE VELOCITY

Consider the straight-crested waves propagating in the x-direction. The fluid loading term is not needed for the phase velocity equation. We assume the transverse motion of the plate as

$$w = A_0 e^{i(kx - kct)} \quad . \tag{30}$$

After substituting (30) into (29) we expand and rearrange terms to obtain the phase velocity equation:

$$\xi^8 \left\{ \frac{1}{36} T^4 (t_1 R_1 + t_2 R_2) (R_1{}^3 R_2{}^2 t_1 + R_1{}^2 R_2{}^3 t_2) \right\}$$

$$+ \xi^6 \Bigg\{ -\frac{1}{36} T^4 (t_1 R_1 + t_2 R_2) \left[(\beta_3{}^2 + 2\beta_2{}^2) R_1{}^3 R_2{}^2 t_1 + (\beta_2{}^2 + 2\beta_3{}^2) R_1{}^2 R_2{}^3 t_2 \right]$$

$$- \frac{1}{36} T^4 (t_1 R_1 + t_2 R_2 \beta_1{}^2) \left[R_1{}^3 R_2{}^2 t_1 \kappa + R_1{}^2 R_2{}^3 t_2 \kappa \right]$$

$$- \frac{1}{3} T^2 \kappa \ (t_1 R_1 + t_2 R_2) \left[R_1 R_2 (R_2 r_1 \beta_1{}^2 t_2 + R_1 r_2 t_1) \right.$$

$$+ \frac{1}{4} t_1 R_1{}^3 \beta_1{}^2 + \frac{1}{4} t_2 R_2{}^3 \Bigg] \Bigg\}$$

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$$+ \xi^4 \left\{ \frac{1}{36} T^4 (t_1 R_1 + t_2 R_2) \left[R_1{}^3 R_2{}^2 t_1 (\beta_2{}^4 + 2\beta_2{}^2 \beta_3{}^2) \right. \right.$$

$$\left. + R_1{}^2 R_2{}^3 t_2 (\beta_3{}^4 + 2\beta_2{}^2 \beta_3{}^2) \right]$$

$$+ \frac{1}{36} T^4 (t_1 R_1 + t_2 R_2 \beta_1{}^2) \left[R_1{}^3 R_2{}^2 t_1 \kappa (\beta_3{}^2 + 2\beta_2{}^2) \right.$$

$$\left. + R_1{}^2 R_2{}^3 t_2 \kappa (\beta_2{}^2 + 2\beta_3{}^2) \right]$$

$$+ \frac{1}{3} T^2 (t_1 R_1 + t_2 R_2) \left[R_1 R_2 \kappa (\beta_2{}^2 + \beta_3{}^2) (R_2 r_1 \beta_1{}^2 t_2 + R_1 r_2 t_1) \right.$$

$$\left. + \frac{1}{2} \beta_2{}^2 t_1 R_1{}^3 \kappa \beta_1{}^2 + \frac{1}{2} \beta_3{}^2 t_2 R_2{}^3 \kappa \right]$$

$$+ \frac{1}{4} T^2 \kappa^2 \beta_1{}^2 \left[\frac{1}{3} t_1{}^2 R_1{}^4 + \frac{1}{3} t_2{}^2 R_2{}^4 + t_1 t_2 R_1{}^2 R_2{}^2 \left(\frac{4}{3} r_1 + \frac{4}{3} r_2 + 2 \right) \right]$$

$$\left. + \kappa^2 \beta_1{}^2 (t_1 R_1 + t_2 R_2) (t_1 R_1 + t_2 R_2) \right\}$$

$$+ \xi^2 \left\{ - \frac{1}{36} T^4 (t_1 R_1 + t_2 R_2) \left[R_1{}^3 R_2{}^2 t_1 \beta_2{}^4 \beta_3{}^2 + R_1{}^2 R_2{}^3 t_2 \beta_2{}^2 \beta_3{}^4 \right] \right.$$

$$- \frac{1}{36} T^4 (t_1 R_1 + t_2 R_2 \beta_1{}^2) \left[R_1{}^3 R_2{}^2 t_1 \kappa (\beta_2{}^4 + 2\beta_2{}^2 \beta_3{}^2) \right.$$

$$\left. + R_1{}^2 R_2{}^3 t_2 \kappa (\beta_3{}^4 + 2\beta_2{}^2 \beta_3{}^2) \right]$$

$$- \frac{1}{3} T^2 \kappa (t_1 R_1 + t_2 R_2) \left[R_1 R_2 (R_2 r_1 \beta_1{}^2 t_2 + R_1 r_2 t_1) \beta_2{}^2 \beta_3{}^2 \right.$$

$$\left. + \frac{1}{4} R_1{}^3 t_1 \beta_2{}^4 \beta_1{}^2 + \frac{1}{4} R_2{}^3 t_2 \beta_3{}^4 \right]$$

$$- \frac{1}{4} T^2 \kappa^2 \beta_1{}^2 \left[\frac{2}{3} \beta_2{}^2 t_1{}^2 R_1{}^4 + \frac{2}{3} \beta_3{}^2 t_2{}^2 R_2{}^4 \right.$$

$$\left. + (\beta_2{}^2 + \beta_3{}^2) t_1 t_2 R_1{}^2 R_2{}^2 \left(\frac{4}{3} r_1 + \frac{4}{3} r_2 + 2 \right) \right]$$

$$- \kappa^2 \beta_1^2 (t_1 R_1 + t_2 R_2)(\beta_2^2 t_1 R_1 + \beta_3^2 t_2 R_2) \Bigg|$$

$$+ \left\{ \frac{1}{36} T^4 (t_1 R_1 + t_2 R_2 \beta_1^2) \left[R_1^3 R_2^2 t_1 \kappa (\beta_2^4 \beta_3^2) + R_1^2 R_2^3 t_2 \kappa (\beta_2^2 \beta_3^4) \right] \right.$$

$$+ \frac{1}{12} T^2 \kappa^2 \beta_1^2 (t_1^2 R_1^4 \beta_2^4 + t_2^2 R_2^4 \beta_3^4)$$

$$+ \frac{1}{4} T^2 t_1 t_2 R_1^2 R_2^2 \kappa^2 \beta_1^2 \beta_2^2 \beta_3^2 \left(\frac{4}{3} r_1 + \frac{4}{3} r_2 + 2 \right) \Bigg\} = 0 \quad , \qquad (31)$$

where

$$\xi^2 = c^2 / c_{s1}^2 \ , \ \beta_1^2 = c_{s2}^2 / c_{s1}^2 \ ,$$

$$\beta_2^2 = c_{p1}^2 / c_{s1}^2 \ , \ \beta_3^2 = c_{p2}^2 / c_{s1}^2$$

$$r_1 = \frac{h_1}{h_2} , \ r_2 = \frac{h_2}{h_1}$$

$$h_1 + h_2 = H \ , \ R_1 = \frac{h_1}{H} \ , \ R_2 = \frac{h_2}{H}$$

$$T = kH,$$

$$t_1 = \frac{\rho_1}{\rho_1 + \rho_2} \quad , \quad t_2 = \frac{\rho_2}{\rho_1 + \rho_2}$$

and

$c = \omega/k$ = phase velocity of the straight-crested waves

c_{s1}, c_{s2} = shear wave velocity $c_{si} = (\mu_i/\rho_i)^{1/2}$ in layer 1 and layer 2

c_{p1}, c_{p2} = longitudinal plate wave velocity $c_{pi} = [E_i/\rho_i (1 - \nu_i^2)]^{1/2}$ in layer 1 and layer 2.

(31) is the equation of the phase velocity. Examination shows four branches of the phase velocity curves versus T, the wavenumber-thickness product. It is readily verified that when the two layers are identical, (31) becomes the equation for a single plate of the Timoshenko-Mindlin model.

NUMERICAL RESULTS

The numerical results of phase velocity curves were computed for a component plate of a steel layer (layer 1) and a copper layer (layer 2). The elastic properties of these layers are as follows:

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Steel

$$E_1 = 2.168663 \times 10^{11} \text{ N/m}^2$$

$$\nu_1 = 0.283629$$

$$\rho_1 = 7.8 \times 10^3 \text{ kg/m}^3$$

$$c_{p1} = 5498.7 \text{ m/s}$$

$$c_{s1} = 3290.9 \text{ m/s}$$

$$c_{R1} = 3044.9 \text{ m/s}$$

Copper

$$E_2 = 1.22 \times 10^{11} \text{ N/m}^2$$

$$\nu_2 = 0.33$$

$$\rho_2 = 8.9 \times 10^3 \text{ kg/m}^3$$

$$c_{p2} = 3922.1 \text{ m/s}$$

$$c_{s2} = 2270.1 \text{ m/s}$$

$$c_{R2} = 2115.8 \text{ m/s}$$

c_{Ri} are the Rayleigh wave velocities.

The normalized phase velocity $\xi = c/c_{s1}$ is plotted versus the wavenumber-thickness product, $T = kH$. Appendix A shows the resulting curves for nine thickness ratios $R_1 = h_1/H$. The following comments pertain to those curves.

As T increases from zero, curves 1 (first antisymmetric mode of motion) increase rapidly until T = 2 and gradually reach an asymptotic value lying within the interval $c_{R2}/c_{s1} - c_{R1}/c_{s1}$ and increasing with thickness ratio R_1. This can be explained easily on the following physical ground. For the limiting case of a single layer, curves 1 attain the Rayleigh wave speed for large T; for the case of two layers, the asymptotic value of curves 1 tend to reach the Rayleigh wave speed of the thicker layer. Furthermore, the shape of curves 1 is very similar to that of the exact solution (ref 3).

Curves 2, representing the phase velocity of the second antisymmetric mode of motion, reach asymptotically the plate wave speed of the soft layer, in this case 1.19. When T = 0, the starting points of these curves increase from c_{p2}/c_{s1} to c_{p1}/c_{s1}, or from the plate wave speed in copper to that in steel as R_1 increases from 0.1 to 0.9. Similarly, curves 4, representing the phase velocity of the fourth antisymmetric mode of motion, reach asymptotically the plate wave speed of the hard layer – in this case steel, with $c_{p1}/c_{s1} = 1.67$. The asymptotic values of curves 2 and 4 do not change with varying R_1.

Curves 3, representing the phase velocity of the third antisymmetric mode of motion, are seen to wander within the band delimited by curves 2 and 4. The asymptotic values of curves 3 behave similarly to those of curves 1, shifting toward the plate wave velocity of the thicker layer as R_1 increases from 0.1 to 0.9.

CONCLUSIONS

An approximate theory of motion of a two-layered plate of elastic, isotropic, and homogeneous materials was developed. Shear and rotary inertia corrections were included in the theory. Equations of motion and of the phase velocities of straight-crested waves were derived. In the limiting case, these equations were shown to be identical to those of single-layered plate. Numerical results showed that dispersive curves of the phase velocities versus the wavenumber-thickness product of the first four antisymmetric modes of motion behave very similarly to those of the exact solution developed by Jones (ref 3). The phase velocity of the lowest mode reaches an asymptotic value lying within the individual Rayleigh wave speeds of the two layers. The asymptotic values of the phase velocities of the second and the fourth modes are recognized as the plate wave speeds of the two layers. The curve representing the third mode is seen to wander within those asymptotic values of the second and fourth modes.

RECOMMENDATIONS

1. Employ this model to investigate the behavior of the scattered signal (target strength) from a two-layered plate. A useful approach is to assume the plate to be loaded with water on one side and air or water on the other side, then to compute and plot the transmission and reflection coefficients of an incident acoustic wave. The procedure for performing the calculations is very similar to that of Graff et al (ref 7). The variation of the coefficients versus the angle of incidence and the wavenumber-thickness product yields information about target strength and its variation at various modes of plate vibration. Furthermore, since the coefficients can be computed from the exact theory, a quantitative comparison can be made and the validity of this approximate model can be examined.

2. Apply this model to a composite plate of one elastic layer (base plate) and one viscoelastic layer (coating layer), to calculate target strength. Impedance discontinuities along the plate such as stiffening ribs and varying thickness can be handled by appropriately modifying the model.

3. Note that this model can also be used where the viscoelastic layer contains many inclusions and/or cavities, provided the "equivalent effective elastic properties" of the viscoelastic layer are calculated and used.

4. If experience with a single-layered plate is any indicator, this model should be improved to provide better resolution of the scattered signal of acoustic waves at grazing incident angles, possibly by combining it with the "Lyamshev Theory" on symmetric modes of vibration.

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[7] Graff, KF, CA Klein, and RG Kouyoumjian, Technical Report 4720-2, A Study of Several Approximate Theories for Calculating the Reflection of Acoustic Plane Waves from Elastic Plates, prepared for NOSC under Contract N66001-77-C-0195, Ohio State University ElectroScience Laboratory, Columbus OH, 15 December 1978.

REFERENCES

1. Timoshenko, SP, On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars, Philosophical Magazine, vol 41, p 744-746, 1921.

2. Mindlin, RD, Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates, J of Applied Mech, vol 18, p 31-38, 1951.

3. Jones, JP, Wave Propagation in a Two-Layered Medium, J of Applied Mech, vol 31, p 213-222, 1964.

4. Lai, JL, Pressure Radiations from an Infinite Two-Layered Elastic Plate with Point and Shear Force Excitations, J of Acoustic Soc of America, vol 53, p 486-494, 1973.

5. Yu, YY, A New Theory of Elastic Sandwich Plates—One Dimensional Case, J of Applied Mech, vol 26, p 415-421, 1959.

6. Mindlin, RD, and MA Medick, Extensional Vibrations of Elastic Plates, J of Applied Mech, vol 26, p 561-569, 1959.

7. Graff, KF, CA Klein, and RG Kouyoumjian, Technical Report 4720-2, A Study of Several Approximate Theories for Calculating the Reflection of Acoustic Plane Waves from Elastic Plates, prepared for NOSC under Contract N66001-77-C-0195, Ohio State University ElectroScience Laboratory, Columbus OH, 15 December 1978.

APPENDIX A: PLOTS OF NORMALIZED PHASE VELOCITY VS WAVENUMBER-THICKNESS PRODUCT

Each figure consists of four dispersive curves of dimensionless phase velocity versus the wavenumber-thickness product, for a given thickness ratio $R_1 = h_1/H$. The four curves are labeled 1–4 to represent the first through the fourth antisymmetric mode of motion.
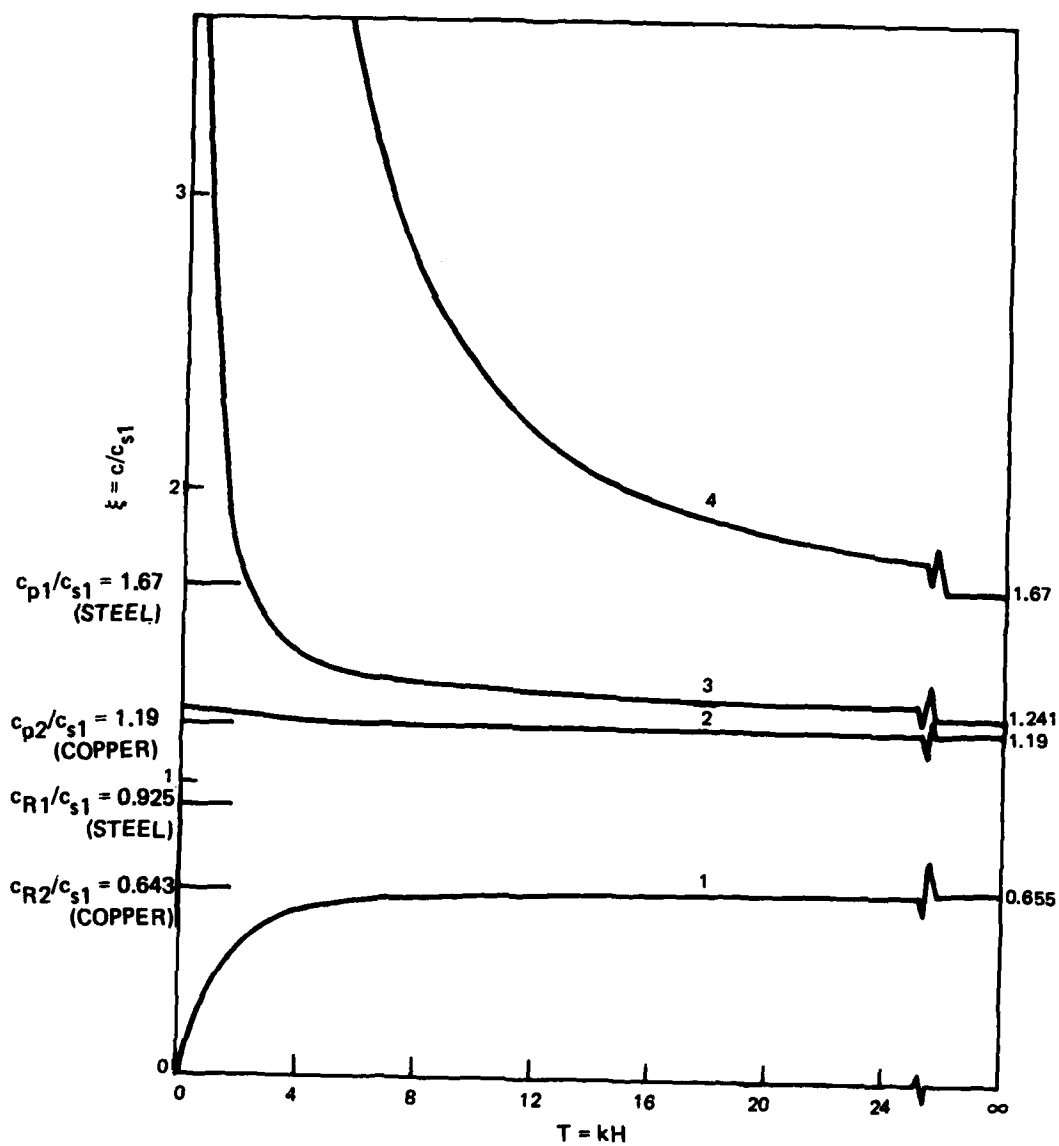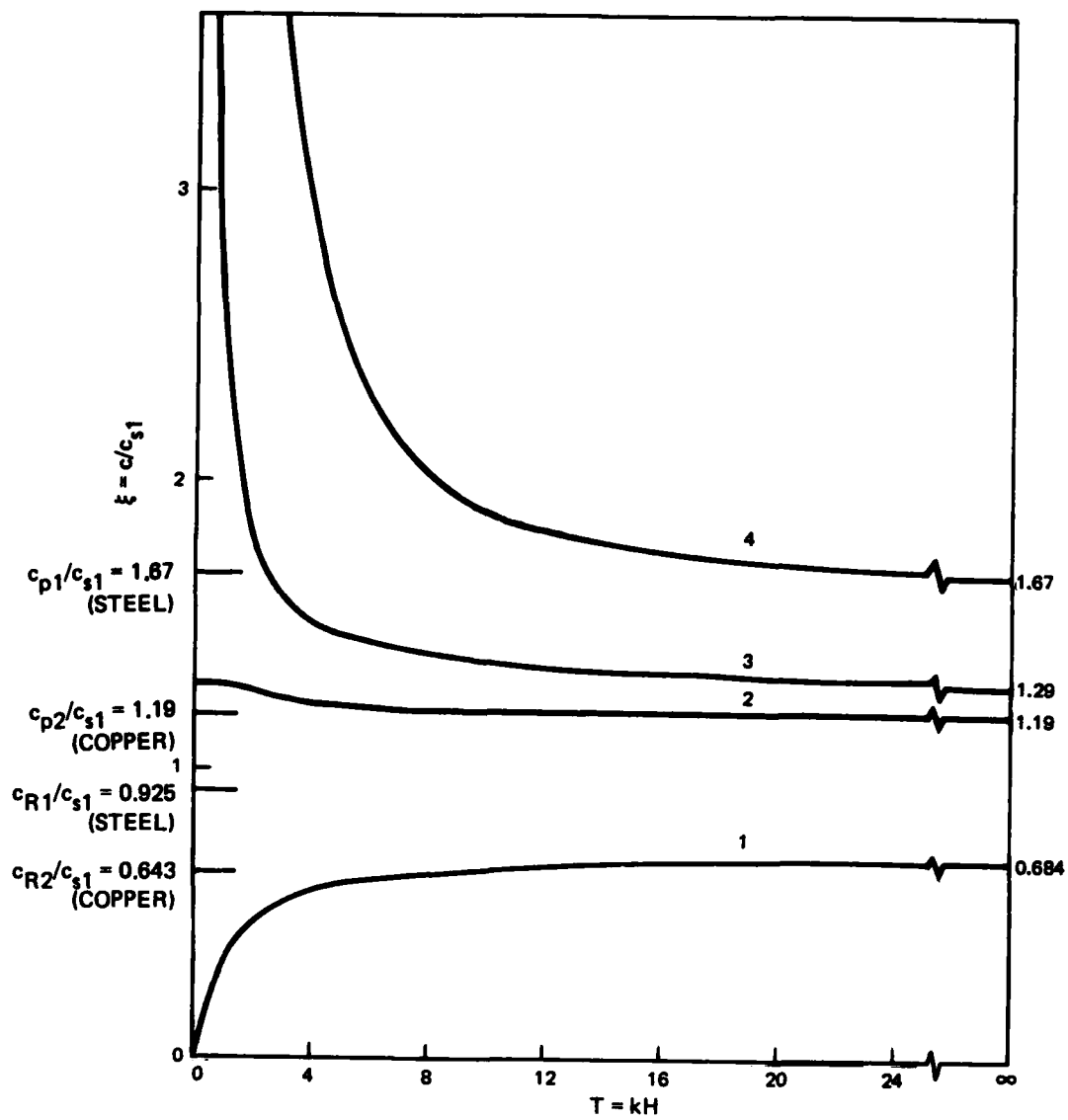
Figure A1. $R_1 = 0.1$.
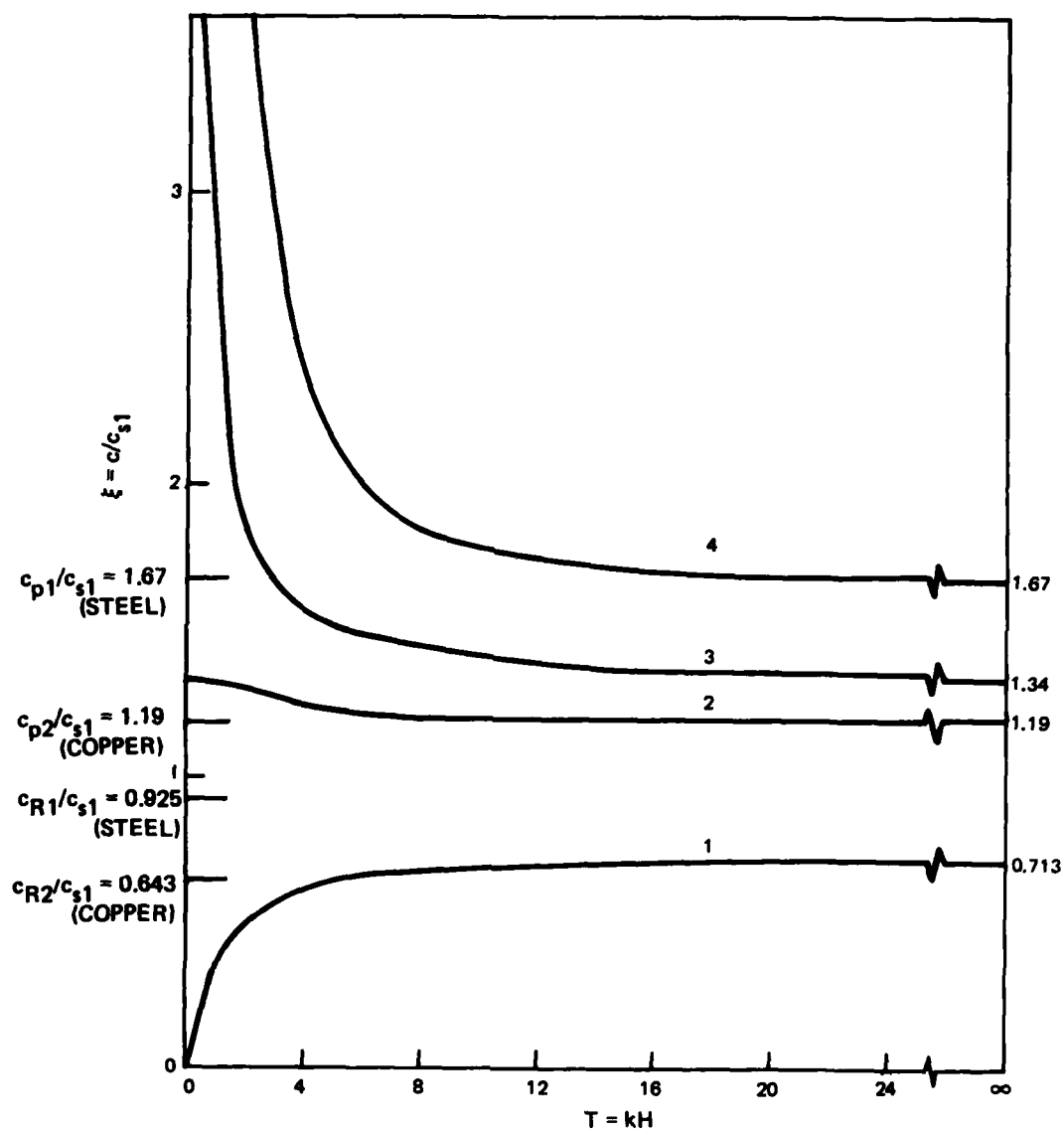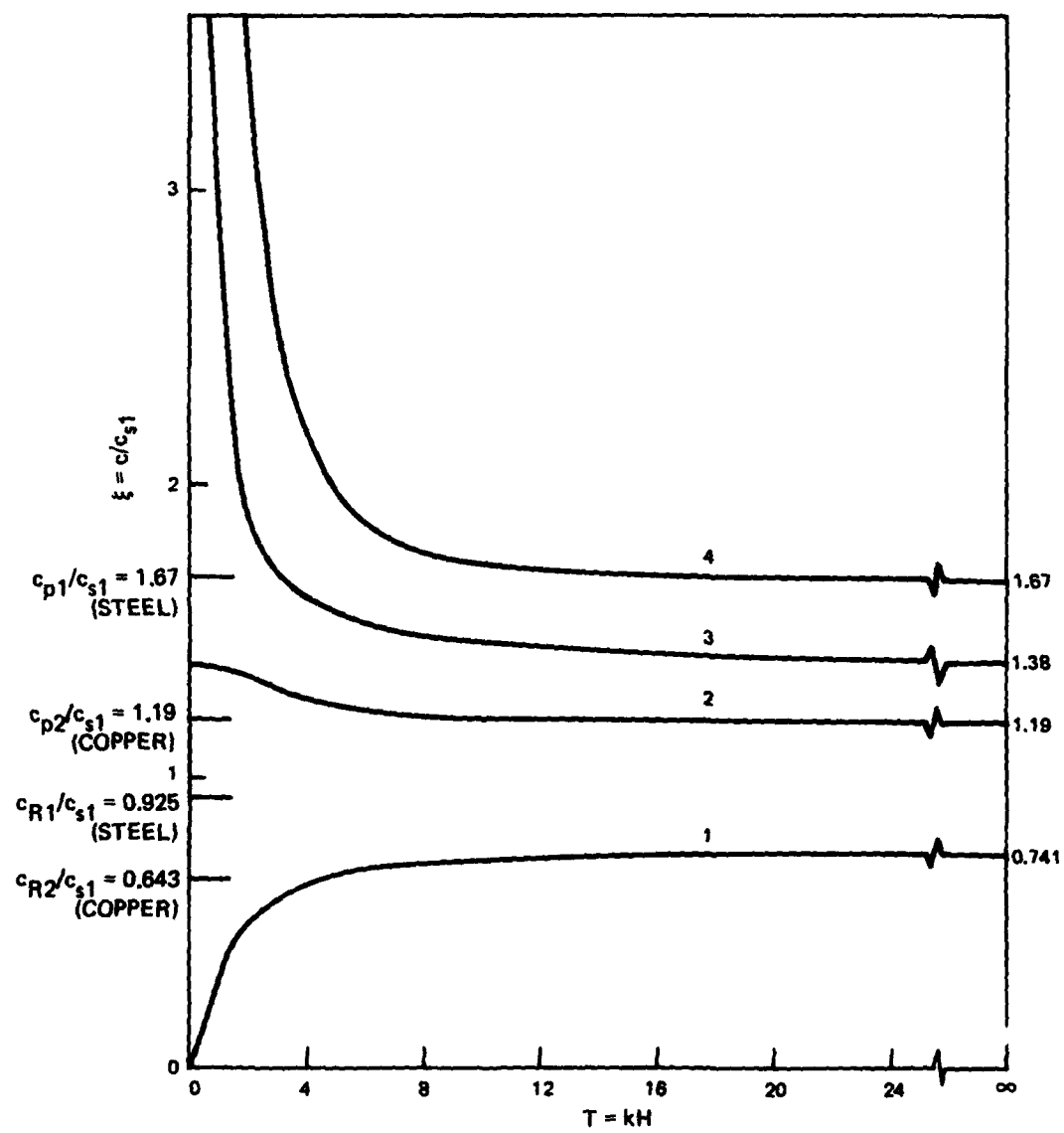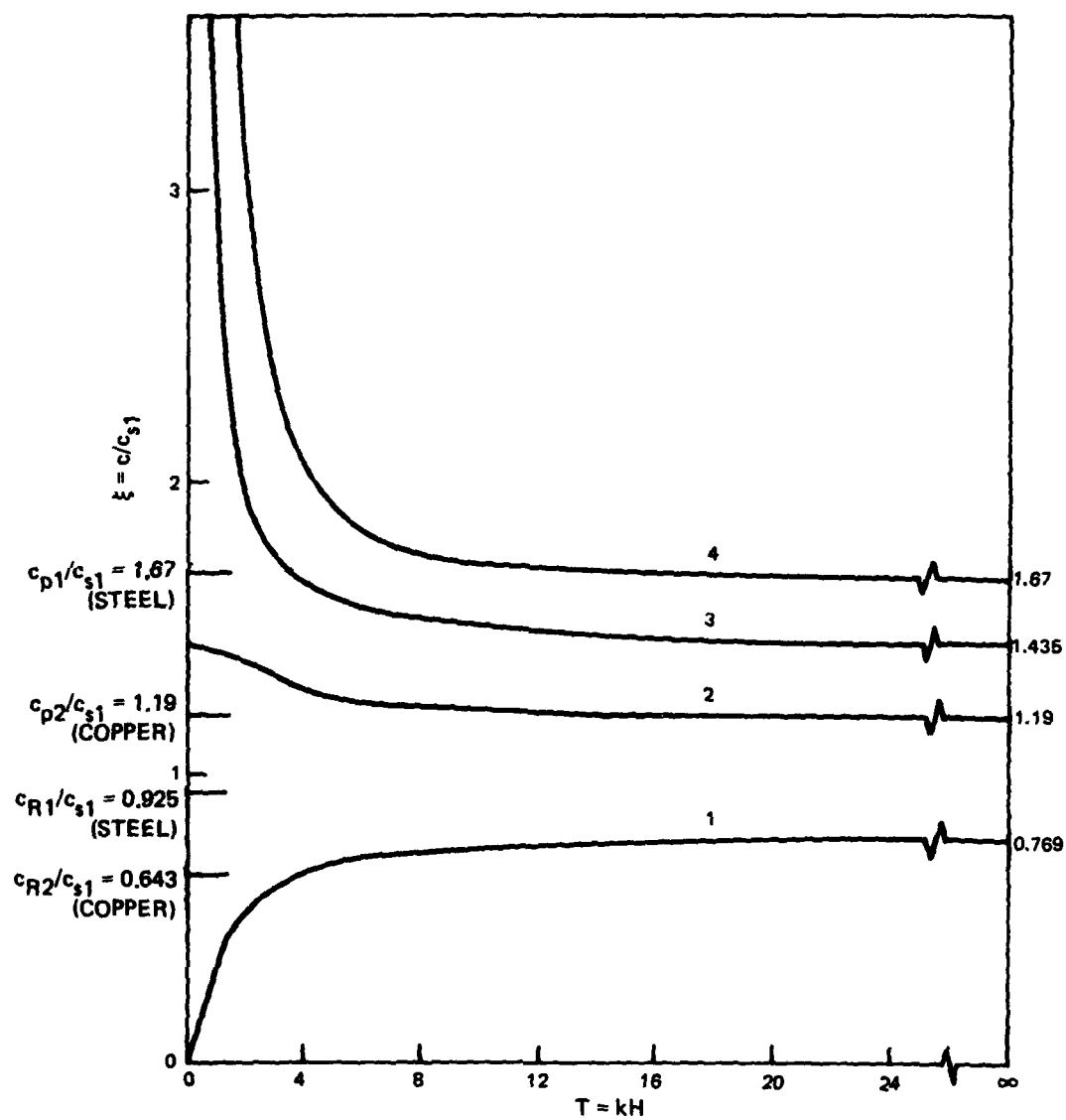
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Figure A2. $R_1 = 0.2$.

Figure A3. $R_1 = 0.3$.

Axis labels and annotations:

$\xi = c/c_{s1}$ (vertical axis)

$c_{p1}/c_{s1} = 1.67$ (STEEL)

$c_{p2}/c_{s1} = 1.19$ (COPPER)

$c_{R1}/c_{s1} = 0.925$ (STEEL)

$c_{R2}/c_{s1} = 0.643$ (COPPER)

$T = kH$ (horizontal axis)

Right-side curve labels: 1.67, 1.34, 1.19, 0.713

Curve numbers: 1, 2, 3, 4

Figure A4. $R_1 = 0.4$.

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Figure A5. $R_1 = 0.5$.

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Figure A6. $R_1 = 0.6$.

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Figure A7. $R_1 = 0.7$.

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Figure A8. $R_1 = 0.8$.

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Figure A9. $R_1 = 0.9$.

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